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# Bases in Banach spaces of smooth functions on Cantor-type sets

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## Abstract

We suggest a Schauder basis in Banach spaces of smooth functions and traces of smooth functions on Cantor-type sets. In the construction, local Taylor expansions of functions are used.

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## 1. Introduction

We consider the basis problem for Banach spaces of differentiable functions. It is not difficult to present a (Schauder) basis in the space  $C^p[0, 1]$ . Indeed, by means of the operator  $T : C[0, 1] \longrightarrow C_{\mathcal{F}}^p[0, 1] : f \mapsto \int_0^x \int_0^{x_1} \cdots \int_0^{x_{p-1}} f(x_p) dx_p \cdots dx_1$  we have an isomorphism  $C^p[0, 1] \simeq \mathbb{R}^p \oplus C[0, 1]$ . Here  $C_{\mathcal{F}}^p[0, 1]$  denotes the subspace of functions that are flat at 0, that is such that  $g^{(k)}(0) = 0$  for  $0 \leq k \leq p - 1$ . Therefore, any Schauder basis in  $C[0, 1]$  gives a corresponding basis in the space  $C^p[0, 1]$ .

For other compact sets  $K$ , the question about a basis in the space  $C^p(K)$  may be much more difficult. For example, one of the basis problems of Banach concerning the space  $C^1[0, 1]^2$  (see [1, p.147]) was solved only 37 years later by Ciesielski in [3] and Schonefeld in [14]. Even after this, a generalization to the case  $C^p[0, 1]^2$  with  $p \geq 2$  was not trivial (see [15] for details).

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Schauder bases in the spaces  $C^p[0, 1]^q$  were suggested independently by Ciesielski and Domsta in [4] and by Schonefeld in [15]. We should notice that two main approaches in the construction of bases were presented in these papers. Schonefeld's system is interpolating basis, while the basis constructed in [4] is orthonormal, but not interpolating.

Mitjagin established in [13, Th.3] that if  $M_1$  and  $M_2$  are  $n$ -dimensional smooth manifolds with or without boundary, then the spaces  $C^p(M_1)$  and  $C^p(M_2)$  are isomorphic. This result essentially enlarges the class of compact sets  $K$  with a basis in the space  $C^p(K)$ , but it cannot be applied to compact sets with infinitely many components, in particular for nontrivial totally disconnected sets.

Jonsson considered in [9] triangulations of compact sets in  $\mathbb{R}$  and constructed an interpolating Schauder basis in the space  $C^p(K)$  provided the compact set  $K$  admits a sequence of regular triangulations. By Theorem 1 in [9], the last condition is valid if and only if  $K$  preserves the so-called Local Markov Inequality, which in turn means that  $K$  is uniformly perfect [11, Section 2.2]. On the other hand, the space considered in [9] was actually  $\mathcal{E}^p(K)$ , that is the Whitney space of functions on  $K$  extendable to functions from  $C^p(\mathbb{R})$ , but equipped with the norm of the space  $C^p(K)$ . It should be noted that, in general, the space  $\mathcal{E}^p(K)$  is not complete in this norm (see [9, p.54] and Section 3).

Here we consider the case of a Cantor-type set  $K$  and present explicitly a Schauder basis in the Banach space  $C^p(K)$  of  $p$  times differentiable on  $K$  functions as well as in the Whitney space  $\mathcal{E}^p(K)$ . In the construction local Taylor expansions of functions are used. In a sense, this generalizes the basis from Haar functions in the space  $C(K)$  for the Cantor set  $K$  [16, Prop. 2.2.5]. Clearly, the system of monomials cannot form a basis in the space  $C^p[0, 1]$  with  $p \leq \infty$ , containing non-analytic functions. In our case, for a Cantor-type set  $K$ , “local Taylor” bases are presented only in the Banach spaces  $\mathcal{E}^p(K)$  with  $p < \infty$ , but not in the Fréchet spaces  $\mathcal{E}(K)$  of Whitney functions of infinite order. For the last case, a basis was suggested in [6] by means of local Newton interpolations; see also [7] for a similar basis in  $C(K)$ . Interpolating Schauder bases in other functional Banach spaces on fractals were given in [10]. It should be noted that not all functional spaces possess interpolating bases [8].

## 2. Local Taylor expansions on Cantor-type sets

Given compact set  $K \subset \mathbb{R}$ ,  $f = (f^{(k)})_{0 \leq k \leq n} \in \prod_{0 \leq k \leq n} C(K)$  and  $a, x \in K$ , let us consider the formal Taylor polynomial  $T_a^n f(x) = \sum_{0 \leq k \leq n} f^{(k)}(a) \frac{(x-a)^k}{k!}$  and the corresponding Taylor remainder  $R_a^n f(x) = f(x) - T_a^n f(x)$ . In the case of perfect  $K$ , the set  $(f^{(k)}(x))_{0 \leq k \leq n, x \in K}$  is completely defined by the values of  $f$  on  $K$  provided existence of the corresponding derivatives. If  $m \leq n$  and  $a, b, c \in K$  then trivially

$$T_a^n \circ T_b^m = T_b^m, \quad R_a^n \circ R_b^m = R_a^n, \quad R_a^n \circ T_b^m = 0. \quad (1)$$

Let  $\Lambda = (l_s)_{s=0}^\infty$  be a sequence such that  $l_0 = 1$  and  $0 < 2l_{s+1} < l_s$  for  $s \in \mathbb{N}_0 := \{0, 1, \dots\}$ . Let  $K(\Lambda)$  be the Cantor set associated with the sequence  $\Lambda$  that is  $K(\Lambda) = \bigcap_{s=0}^\infty E_s$ , where  $E_0 = I_{1,0} = [0, 1]$ ,  $E_s$  is a union of  $2^s$  closed *basic* intervals  $I_{j,s} = [a_{j,s}, b_{j,s}]$  of length  $l_s$  and  $E_{s+1}$  is obtained by deleting the open concentric subinterval of length  $h_s := l_s - 2l_{s+1}$  from each  $I_{j,s}$ ,  $j = 1, 2, \dots, 2^s$ .

Let us consider the set of all left endpoints of basic intervals. Since  $a_{j,s} = a_{2j-1,s+1}$  for  $j \leq 2^s$ , any such point has infinitely many representations in the form  $a_{j,s}$ . We select the representation with the minimal second subscript and call it the *minimal representation*. If  $j$

is even, then the representation  $a_{j,s}$  is minimal for the corresponding point. Otherwise, for  $j = 2^q(2m+1) + 1 > 1$  we obtain  $a_{j,s} = a_{2m+2,s-q}$ . Clearly,  $a_{1,s} = a_{1,0}$  for all  $s$ . Therefore we have a bijection between the set of all left endpoints of basic intervals and the set  $A = a_{1,0} \cup (a_{2j,s})_{j=1,s=1}^{2^{s-1},\infty}$ .

Let us enumerate the set  $A$  by first increasing  $s$ , then  $j$ :  $x_1 = a_{1,0} = 0$ ,  $x_2 = a_{2,1} = 1 - l_1$ ,  $x_3 = a_{2,2} = l_1 - l_2$ ,  $x_4 = a_{4,2} = 1 - l_2$ , ... and, in general,  $x_{2^s+k} = a_{2k,s+1}$  for  $k = 1, 2, \dots, 2^s$ .

Let us fix  $p \in \mathbb{N}$ . For  $s \in \mathbb{N}_0$ ,  $j \leq 2^s$  and  $0 \leq k \leq p$  let  $e_{k,j,s}(x) = (x - a_{j,s})^k/k!$  if  $x \in K(\Lambda) \cap I_{j,s}$  and  $e_{k,j,s} = 0$  on  $K(\Lambda)$  otherwise. Given  $f = (f^{(k)})_{0 \leq k \leq p} \in \prod_{0 \leq k \leq p} C(K(\Lambda))$ , let  $\xi_{k,j,s}(f) = f^{(k)}(a_{j,s})$  for the same values of  $s, j$ , and  $k$  as above. Clearly, for the fixed level  $s$ , the system  $(e_{k,j,s}, \xi_{k,j,s})$  is biorthogonal, that is  $\xi_{k,j,s}(e_{n,i,s}) = \delta_{kn} \cdot \delta_{ij}$ . In order to obtain biorthogonality as well with regard to  $s$ , we will use the following convolution property of the values of functionals on the basis elements (see [5, L.3.1] and [6, L.2]). Let  $I_{i,n} \supset I_{j,s-1}$ . Then

$$\sum_{m=k}^p \xi_{k,2j,s}(e_{m,j,s-1}) \cdot \xi_{m,j,s-1}(e_{q,i,n}) = \xi_{k,2j,s}(e_{q,i,n}) \quad \text{for all } q \leq p.$$

Indeed,  $(e_{k,i,n})_{k=0}^p, (e_{k,j,s-1})_{k=0}^p, (e_{k,2j,s})_{k=0}^p$  are three bases in the space  $\mathcal{P}_p(I_{2j,s})$  of polynomials of degree not greater than  $p$  on the interval  $I_{2j,s}$ . If  $M_{r \leftarrow t}$  denotes the transition matrix from the  $t$ -th basis to the  $r$ -th basis, then the identity above means  $M_{3 \leftarrow 2} M_{2 \leftarrow 1} = M_{3 \leftarrow 1}$ .

On the other hand, in our case, this identity is the corresponding binomial expansion:

$$\sum_{m=k}^q \frac{(a_{2j,s} - a_{j,s-1})^{m-k}}{(m-k)!} \cdot \frac{(a_{j,s-1} - a_{i,n})^{q-m}}{(q-m)!} = \frac{(a_{2j,s} - a_{i,n})^{q-k}}{(q-k)!}.$$

Here we consider summation until  $q$  since for  $q < m \leq p$ , the terms  $\xi_{m,j,s-1}(e_{q,i,n})$  vanish.

We restrict our attention only to the functions  $(e_{k,1,0})_{k=0}^p$  and  $(e_{k,2j,s})_{k=0}^{p, 2^{s-1}, \infty}$  corresponding to the set  $A$ . Let us enumerate this family in the lexicographical order with respect to the triple  $(s, j, k)$ :  $f_n = e_{n-1,1,0} = \frac{1}{(n-1)!}(x - x_1)^{n-1} \cdot \chi_{1,0}$  for  $n = 1, 2, \dots, p+1$ . Here and in what follows,  $\chi_{j,s}$  denotes the characteristic function of the interval  $I_{j,s}$ . After this,  $f_n = e_{n-p-2,2,1} = \frac{1}{(n-p-2)!}(x - x_2)^{n-p-2} \cdot \chi_{2,1}$  for  $n = p+2, p+3, \dots, 2(p+1)$  and in general, if  $(m-1)(p+1) + 1 \leq n \leq m(p+1)$ , then  $f_n = \frac{1}{k!}(x - x_m)^k \cdot \chi_{2i,s+1} = e_{k,2i,s+1}$ . Here  $m = 2^s + i$  with  $1 \leq i \leq 2^s$  and  $k = n - (m-1)(p+1) - 1$ . We see that all functions of the type  $\frac{1}{k!}(x - x_m)^k \cdot \chi_{2i,s+1}$  with  $0 \leq k \leq p$  and  $m = 2^s + i \in \mathbb{N}$  are included into the sequence  $(f_n)_{n=1}^\infty$ .

For the same values of parameters as above, we define the functionals  $\eta_{k,1,0} = \xi_{k,1,0}$  for  $k = 0, 1, \dots, p$  and

$$\eta_{k,2j,s} = \xi_{k,2j,s} - \sum_{m=k}^p \xi_{k,2j,s}(e_{m,j,s-1}) \cdot \xi_{m,j,s-1}$$

for  $s \in \mathbb{N}$ ,  $j = 1, 2, \dots, 2^{s-1}$ , and  $k = 0, 1, \dots, p$ . In what follows, we will use the minimal representations of the points  $a_{j,s}$  and the corresponding functionals  $\xi_{m,j,s}$ . For example,  $\eta_{k,2,s} = \xi_{k,2,s} - \sum_{m=k}^p \xi_{k,2,s}(e_{m,1,0}) \cdot \xi_{m,1,0}$ . This agreement is justified by the fact that the value  $\xi_{m,j,s}(f) = f^{(m)}(a_{j,s})$  does not depend on the representation of the point  $a_{j,s}$  and the functions  $e_{m,j,s-1}, e_{m,r,s-q}$  coincide on the interval  $I_{2j,s}$  if  $a_{j,s-1} = a_{r,s-q}$ .

The crucial point of the construction is that the functionals  $\eta_{k,2j,s}$  are biorthogonal, not only to all elements  $(e_{k,2j,s-1})_{k=0}^p$ , but also, by the convolution property, to all  $(e_{k,2i,n})_{k=0}^p$  with  $n = 0, 1, \dots, s-2$  and  $i = 1, 2, \dots, 2^{n-1}$ . In addition, the functional  $\eta_{k,2j,s}$  takes zero value at all elements  $(e_{k,2i,n})_{k=0}^p$  with  $n \geq s$ , except  $e_{k,2j,s}$ , where it equals 1.

In the same lexicographical order as above, we arrange all functionals  $(\eta_{k,1,0})_{k=0}^p$  and  $(\eta_{k,2j,s})_{k=0, j=1, s=1}^{p, 2^{s-1}, \infty}$  into the sequence  $(\eta_n)_{n=1}^\infty$ .

Our next goal is to express the sum  $S_N(f) := \sum_{n=1}^N \eta_n(f) \cdot f_n$  in terms of the Taylor polynomials of the function  $f$ . Clearly,  $S_N(f) = T_0^{N-1} f$  for  $1 \leq N \leq p+1$ .

Suppose  $p+2 \leq N \leq 2(p+1)$ . Then  $S_N(f) = T_0^p f$  on  $I_{1,1}$ . On the interval  $I_{2,1}$ , we obtain  $S_N(f) = T_0^p f + \sum_{n=p+2}^N \eta_{n-p-2,2,1}(f) \cdot e_{n-p-2,2,1}$ . For the second term, we have  $\sum_{k=0}^{N-p-2} [\xi_{k,2,1}(f) - \sum_{m=k}^p \xi_{k,2,1}(e_{m,1,0}) \cdot \xi_{m,1,0}(f)] \frac{1}{k!} (x - a_{2,1})^k = \sum_{k=0}^{N-p-2} \left[ f^{(k)}(a_{2,1}) - \sum_{m=k}^p \frac{1}{(m-k)!} a_{2,1}^{m-k} \cdot f^{(m)}(0) \right] \frac{1}{k!} (x - a_{2,1})^k = \sum_{k=0}^{N-p-2} (R_0^p f)^{(k)}(a_{2,1}) \frac{1}{k!} (x - a_{2,1})^k = T_{a_{2,1}}^{N-p-2}(R_0^p f)$ .

Therefore,  $S_N(f) = T_0^p f$  on  $I_{1,1}$  and  $S_N(f) = T_0^p f + T_{a_{2,1}}^{N-p-2}(R_0^p f)$  on  $I_{2,1}$ . Particularly,  $S_{2p+2}(f) = T_0^p f + T_{a_{2,1}}^p(R_0^p f) = T_{a_{2,1}}^p f$ , by (1). In addition,  $S_N^{(k)}(f)(a_{2,1}) = f^{(k)}(a_{2,1})$  for  $0 \leq k \leq N - p - 2$ , as is easy to check.

Continuing in this way, the values  $2p+3 \leq N \leq 3(p+1)$  correspond to the passage on the interval  $I_{2,2}$  from the polynomial  $T_0^p f$  to the polynomial  $T_{a_{2,2}}^p f$  and the values  $3p+4 \leq N \leq 4(p+1)$  in turn transform  $T_{a_{2,1}}^p f$  on  $I_{4,2}$  into  $T_{a_{4,2}}^p f$ .

By the same argument,  $S_{2^s(p+1)}(f) = T_{a_{j,s}}^p f$  on  $I_{j,s}$  for  $1 \leq j \leq 2^s$  and if  $j$  with  $0 \leq j < 2^s$  is fixed, then the values  $N = 2^s(p+1) + j(p+1) + m+1$  with  $0 \leq m \leq p$  transform  $T_{a_{j+1,s}}^p f$  on  $I_{2j+2,s+1}$  into  $T_{a_{2j+2,s+1}}^p f$ .

Combining all considerations of this section yields the following result:

**Lemma 1.** *The system  $(f_n, \eta_n)_{n=1}^\infty$  is biorthogonal. Given  $f = (f^{(k)})_{0 \leq k \leq p} \in \prod_{0 \leq k \leq p} C(K(A))$  and  $N = 2^s(p+1) + j(p+1) + m+1$  with  $s \in \mathbb{N}_0$ ,  $0 \leq j < 2^s$ , and  $0 \leq m \leq p$  we have  $S_N(f) = T_{a_{k,s+1}}^p f$  on  $I_{k,s+1}$  with  $k = 1, 2, \dots, 2j+1$ ,  $S_N(f) = T_{a_{k,s}}^p f$  on  $I_{k,s}$  with  $k = j+2, j+3, \dots, 2^s$ , and  $S_N(f) = T_{a_{j+1,s}}^p f + T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)$  on  $I_{2j+2,s+1}$ .*

### 3. Spaces of differentiable functions and their traces

Let  $K$  be a compact subset of  $\mathbb{R}$ ,  $p \in \mathbb{N}$ . Then the finite product  $\prod_{0 \leq k \leq p} C(K)$  equipped with the norm  $\|(f^{(k)})_{0 \leq k \leq p}\|_p = \sup\{|f^{(k)}(x)| : x \in K, k \leq p\}$  is a Banach space. We will consider its subspace  $C^p(K)$  consisting of functions on  $K$  such that for every nonisolated point  $x \in K$  there exist continuous derivatives  $f^{(k)}(x)$  of order  $k \leq p$  defined in a usual way. If the point  $x$  is isolated, then the set  $(f^{(k)}(x))_{0 \leq k \leq p}$  can be taken arbitrarily.

The space  $\mathcal{E}^p(K)$  of Whitney functions of order  $p$  consists of functions from  $C^p(K)$  that are extendable to  $C^p$ -functions on  $\mathbb{R}$ . Due to Whitney [18],

$$f = (f^{(k)})_{0 \leq k \leq p} \in \mathcal{E}^p(K) \text{ if}$$

$$(R_y^p f)^{(k)}(x) = o(|x - y|^{p-k}) \quad \text{for } k \leq p \text{ and } x, y \in K \text{ as } |x - y| \rightarrow 0. \quad (2)$$

The natural topology of a Banach space is given in  $\mathcal{E}^p(K)$  by the norm

$$\|f\|_p = \|f\|_p + \sup \left\{ |(R_y^p f)^{(k)}(x)| \cdot |x - y|^{k-p}; x, y \in K, x \neq y, k = 0, 1, \dots, p \right\}.$$

The Fréchet spaces  $C^\infty(K)$  and  $\mathcal{E}(K)$  are obtained as the projective limits of the corresponding sequences of spaces. Similarly, the spaces  $\mathcal{E}^p(K)$ ,  $\mathcal{E}(K)$  can be defined for  $K \subset \mathbb{R}^d$  with  $d > 1$ .

In general, the spaces  $C^p(K)$  and  $C^\infty(K)$  contain nonextendable functions and the norms  $\|f\|_p$  and  $|f|_p$  are not equivalent on  $\mathcal{E}^p(K)$ . A compact set  $K \subset \mathbb{R}^d$  is called Whitney  $r$ -regular if it is connected by rectifiable arcs, and there exists a constant  $C$  such that  $\sigma(x, y)^r \leq C|x - y|$  for all  $x, y \in K$ . Here  $\sigma$  denotes the intrinsic (or geodesic) distance in  $K$ . The case  $r = 1$  gives the Whitney property (P) [19]. If  $K$  is 1-regular, then  $C^p(K) = \mathcal{E}^p(K)$  [19, T.1]. A sufficient condition for coincidence  $C^\infty(K) = \mathcal{E}(K)$  is  $r$ -regularity of  $K$  for some  $r$ . For an estimation of  $\|\cdot\|_p$  by  $|\cdot|_p$  in this case, we refer the reader to [17, IV, 3.11] and [2].

For one-dimensional compact sets we have the following trivial result:

**Proposition 1.**  $C^p(K) = \mathcal{E}^p(K)$  for  $2 \leq p \leq \infty$  if and only if  $K = \bigcup_{n=1}^N [a_n, b_n]$  with  $a_n \leq b_n$  for  $n \leq N$ .

**Proof.** Indeed, if  $K$  is a finite union of closed intervals, then for any  $C^p$ -function on  $K$  there exists a corresponding extension of the same smoothness, and what is more, the extension which is analytic outside  $K$  can be chosen (see e.g. in [12, Cor.2.2.3]).

In the converse case, the complement  $\mathbb{R} \setminus K$  contains infinitely many disjoint open intervals. Therefore there exists at least one point  $c \in K$  which is an accumulation point of these intervals. Let  $K \subset [a, b]$  with  $a, b \in K$ . Without loss of generality we can assume that  $[c, b]$  contains a sequence of intervals from  $\mathbb{R} \setminus K$ . Then  $K \subset K_0 := [a, c] \cup \bigcup_{n=1}^\infty [a_n, b_n]$  with  $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \subset K, b_1 = b, a_{n+1} \leq b_{n+1} < a_n, (b_{n+1}, a_n) \subset \mathbb{R} \setminus K$  for all  $n$ . Given  $1 < p < \infty$ , let us take  $F = 0$  on  $[a, c]$ ,  $F = (a_n - c)^p$  on  $[a_n, b_n]$  if  $a_n < b_n$ . In the case  $a_n = b_n$  let  $F(a_n) = (a_n - c)^p$  and  $F^{(k)}(a_n) = 0$  for all  $k > 1$ . Thus,  $F' \equiv 0$ . Then  $f = F|_K$  belongs to  $C^\infty(K)$ , but is not extendable to  $C^p$ -functions on  $\mathbb{R}$  because of violation of (2) for  $y = c, x = a_n, k = 0$ .  $\square$

This nonextendable function can be easily approximated in  $|\cdot|_p$  by extendable functions. Therefore, by the open mapping theorem, the following is obtained:

**Corollary 1.** If  $1 < p < \infty$  and  $K$  is not a finite union of (maybe degenerated) segments, then the space  $(\mathcal{E}^p(K), |\cdot|_p)$  is not complete. The same result is valid for  $(\mathcal{E}(K), (|\cdot|_p)_{p=0}^\infty)$ .

It is interesting that the case  $p = 1$  is exceptional here.

**Examples.** 1. Let  $K = \{0\} \cup (2^{-n})_{n=1}^\infty$ . Then  $C^1(K) = \mathcal{E}^1(K)$ . Indeed, the function  $f \in C^1(K)$  is defined here by two sequences  $(f_n)_{n=0}^\infty$  and  $(f'_n)_{n=0}^\infty$  with  $\gamma_n := (f_n - f_0) \cdot 2^n - f'_0 \rightarrow 0$  and  $f'_n \rightarrow f'_0$  as  $n \rightarrow \infty$ . The second condition gives (2) with  $k = 1$ . The first condition means (2) with  $k = 0, y = 0$ . For the remaining case  $x = 2^{-n}, y = 2^{-m}$ , we have  $R_y^1 f(x) = f_n - f_m - f'_m(2^{-n} - 2^{-m}) = \gamma_n \cdot 2^{-n} - \gamma_m \cdot 2^{-m} + (2^{-n} - 2^{-m})(f'_0 - f'_m)$ , which is  $o(|2^{-n} - 2^{-m}|)$  as  $m, n \rightarrow \infty$ , since  $\max\{2^{-n}, 2^{-m}\} \leq 2 \cdot |2^{-n} - 2^{-m}|$ . Thus,  $f \in \mathcal{E}^1(K)$ .

2. Let  $K = \{0\} \cup (1/n)_{n=1}^\infty$ ,  $f\left(\frac{1}{2m-1}\right) = 0$ ,  $f\left(\frac{1}{2m}\right) = \frac{1}{m\sqrt{m}}$  for  $m \in \mathbb{N}$ , and  $f' \equiv 0$  on  $K$ . Then  $f \in C^1(K)$ , but by the mean value theorem, there is no differentiable extension of  $f$  to  $\mathbb{R}$ .

#### 4. Schauder bases in the spaces $C^p(K(\Lambda))$ and $\mathcal{E}^p(K(\Lambda))$

Let us show that the biorthogonal system suggested in Section 2 is a Schauder basis in both spaces  $C^p(K(\Lambda))$  and  $\mathcal{E}^p(K(\Lambda))$ . Here, as before,  $p \in \mathbb{N}$ . Given  $g$  on  $K(\Lambda)$ , let  $\omega(g, \cdot)$  be the

modulus of continuity of  $g$ , that is  $\omega(g, t) = \sup\{|g(x) - g(y)| : x, y \in K(\Lambda), |x - y| \leq t\}$ ,  $t > 0$ . If  $x \in I = [a, a + l_s]$ , then for any  $i \leq p$  we have easily

$$|(R_a^p f)^{(i)}(x)| < \omega(f^{(i)}, l_s) + l_s \cdot 2|f|_p \quad (3)$$

and

$$|(R_a^p f)^{(i)}(x)| < 4|f|_p. \quad (4)$$

**Lemma 2.** *The system  $(f_n, \eta_n)_{n=1}^\infty$  is a Schauder basis in the space  $C^p(K(\Lambda))$ .*

**Proof.** Given  $f \in C^p(K(\Lambda))$  and  $\varepsilon > 0$ , we want to find  $N_\varepsilon$  with  $|f - S_N(f)|_p \leq \varepsilon$  for  $N \geq N_\varepsilon$ . Let us take  $S$  such that for all  $i \leq p$  we have

$$3 \cdot \omega(f^{(i)}, l_s) + 14 \cdot l_s \cdot |f|_p < \varepsilon. \quad (5)$$

Set  $N_\varepsilon = 2^S(p+1)$ . Then any  $N \geq N_\varepsilon$  has a representation in the form  $N = 2^s(p+1) + j(p+1) + m + 1$  with  $s \geq S$ ,  $0 \leq j < 2^s$ , and  $0 \leq m \leq p$ . Let us fix  $i \leq p$  and apply Lemma 1 to  $R := (f - S_N(f))^{(i)}(x)$  for  $x \in K(\Lambda)$ .

If  $x \in I_{k,s+1}$  with  $k = 1, \dots, 2j+1$ , then  $|R| = |(R_{a_{k,s+1}}^p f)^{(i)}(x)| < \varepsilon$ , by (3) and (5).

If  $x \in I_{k,s}$  with  $k = j+2, j+3, \dots, 2^s$ , then  $|R| = |(R_{a_{k,s}}^p f)^{(i)}(x)|$  and the same arguments can be used.

Suppose  $x \in I_{2j+2,s+1}$ . Then  $|R| \leq |(R_{a_{j+1,s}}^p f)^{(i)}(x)| + |(T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f))^{(i)}(x)|$ . For the first term we use (3). The addend vanishes if  $m < i$ . Otherwise, it is

$$\left| (R_{a_{j+1,s}}^p f)^{(i)}(x) - (R_{a_{j+1,s}}^p f)^{(i)}(a_{2j+2,s+1}) - \sum_{k=i+1}^m (R_{a_{j+1,s}}^p f)^{(k)}(a_{2j+2,s+1}) \frac{(x - a_{2j+2,s+1})^{k-i}}{(k-i)!} \right|.$$

Here, we estimate the first and the second terms by means of (3). For the remaining sum, we use (4):  $|\sum_{k=i+1}^m \dots| \leq 4|f|_p \sum_{k=i+1}^m l_{s+1}^{k-i}/(k-i)! < l_{s+1} \cdot 8|f|_p$ . Combining these we conclude that  $|R| \leq 3(\omega(f^{(i)}, l_s) + l_s \cdot 2|f|_p) + l_{s+1} \cdot 8|f|_p$ . This does not exceed  $\varepsilon$  due to the choice of  $S$ . Therefore,  $|f - S_N(f)|_p \leq \varepsilon$  for  $N \geq N_\varepsilon$ .  $\square$

The main result is given for Cantor-type sets under mild restriction:

$$\exists C_0 : l_s \leq C_0 \cdot h_s, \quad \text{for } s \in \mathbb{N}_0. \quad (6)$$

**Theorem 3.** *Let  $K(\Lambda)$  satisfy (6). Then the system  $(f_n, \eta_n)_{n=1}^\infty$  is a Schauder basis in the space  $\mathcal{E}^p(K(\Lambda))$ .*

**Proof.** Given  $f \in \mathcal{E}^p(K(\Lambda))$ , we show that the sequence  $(S_N(f))$  converges to  $f$  as well in the norm  $\|\cdot\|_p$ . Because of Lemma 2, we only have to check that  $|(R_y^p(f - S_N(f)))^{(i)}(x)| \cdot |x - y|^{i-p}$  is uniformly small (with respect to  $x, y \in K$  with  $x \neq y$  and  $i \leq p$ ) for large enough  $N$ . Fix  $\varepsilon > 0$ . Due to the condition (2), we can take  $S$  such that

$$|(R_y^p f)^{(k)}(x)| < \varepsilon |x - y|^{p-k} \quad \text{for } k \leq p \text{ and } x, y \in K(\Lambda) \text{ with } |x - y| \leq l_s. \quad (7)$$

As above, let  $N_\varepsilon = 2^S(p+1)$  and  $N = 2^s(p+1) + j(p+1) + m + 1$  with  $s \geq S$ ,  $0 \leq j < 2^s$ , and  $0 \leq m \leq p$ .

For simplicity, we take the value  $i = 0$  since the general case can be analyzed in the same manner. We will consider different positions of  $x$  and  $y$  on  $K(\Lambda)$  in order to show

$$|R_y^p(f - S_N(f))(x)| < C\varepsilon|x - y|^p,$$

where the constant  $C$  does not depend on  $x$  and  $y$ . In all cases, we use the representation of  $S_N(f)$  given in Lemma 1.

Suppose first that  $x, y$  belong to the same interval  $I_{k,s+1}$  with some  $k = 1, \dots, 2j + 1$ . Then  $(f - S_N(f))(x) = R_{a_{k,s+1}}^p f(x)$ . From (1) it follows that  $R_y^p(f - S_N(f))(x) = R_y^p f(x)$ . Here,  $|x - y| \leq l_{s+1}$ , so we have the desired bound by (7).

Similar arguments apply to the case  $x, y \in I_{k,s}$  with  $k = j + 2, j + 3, \dots, 2^s$ .

If  $x, y \in I_{2j+2,s+1}$ , then  $(f - S_N(f))(x) = (R_{a_{j+1,s}}^p f)(x) - T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)(x)$  for  $m < p$  and  $(f - S_N(f))(x) = (R_{a_{2j+2,s+1}}^p f)(x)$  for  $m = p$ . Since  $R^p(T^m) = 0$  for  $m < p$ , in both cases we get  $R_y^p(f - S_N(f))(x) = R_y^p f(x)$  with  $|x - y| \leq l_s$  and (7) can be applied once again.

We now turn to the cases when  $x$  and  $y$  lie on different intervals. Let  $x \in I_{k,s+1}, y \in I_{m,s+1}$  with distinct  $k, m = 1, \dots, 2j + 1$ . Then  $R_y^p(f - S_N(f))(x) = R_{a_{k,s+1}}^p f(x) - \sum_{i=0}^p (R_{a_{m,s+1}}^p)^{(i)} f(y)(x - y)^i / i!$ . Here,  $|x - a_{k,s+1}| \leq l_{s+1}$ , and  $|y - a_{m,s+1}| \leq l_{s+1}$ ; thus, applying (7) gives  $|R_y^p(f - S_N(f))(x)| < \varepsilon \cdot l_{s+1}^p + \varepsilon \cdot \sum_{i=0}^p l_{s+1}^{p-i} |x - y|^i / i!$ . Now,  $|x - y| \geq h_s \geq C_0^{-1} l_s$ , by hypothesis. Therefore,  $|R_y^p(f - S_N(f))(x)| < C_0^p (e + 1) \cdot \varepsilon \cdot |x - y|^p$ , which establishes the desired result. Clearly, the same conclusion can be drawn for  $x \in I_{k,s}, y \in I_{m,s}$  with distinct  $k, m = j + 2, \dots, 2^s$ , as well for the case when one of the points  $x, y$  belongs to  $I_{k,s+1}$  with  $k \leq 2j + 1$  whereas another lies on  $I_{m,s}$  with  $m = j + 2, \dots, 2^s$ .

It remains to consider the most difficult cases: just one of the points  $x, y$  belongs to  $I_{2j+2,s+1}$ . Suppose  $x \in I_{2j+2,s+1}$ . We can assume that  $y \in I_{2j+1,s+1}$  since other positions of  $y$  only enlarge  $|x - y|$ . Here,  $R_y^p(f - S_N(f))(x) = R_{a_{j+1,s}}^p f(x) - T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)(x) - \sum_{i=0}^p (R_{a_{2j+1,s+1}}^p)^{(i)} f(y)(x - y)^i / i!$ . We only need to estimate the intermediate  $T^m$  since other terms can be handled in the same way as above. Now,  $|T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)(x)| \leq \sum_{i=0}^m |(R_{a_{j+1,s}}^p)^{(i)} f(a_{2j+2,s+1})| |x - a_{2j+2,s+1}|^i / i!$ . As before, we use (7). In addition,  $|a_{2j+2,s+1} - a_{j+1,s}|$  and  $|x - a_{2j+2,s+1}|$  do not exceed  $C_0|x - y|$ . By that  $|T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)(x)| \leq C_0^p e\varepsilon|x - y|^p$ .

In the last case  $x \in I_{2j+1,s+1}, y \in I_{2j+2,s+1}$ , we have  $R_y^p(f - S_N(f))(x) = R_{a_{j+1,s}}^p f(x) - \sum_{i=0}^p [R_{a_{j+1,s}}^p f - T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)]^{(i)}(y)(x - y)^i / i!$ . As above, it is sufficient to consider only  $\sum_{i=0}^p [T_{a_{2j+2,s+1}}^m (R_{a_{j+1,s}}^p f)]^{(i)}(y)(x - y)^i / i!$  since for other terms we have the desired bound. Of course, the genuine summation here is until  $i = m$ . Let us consider a typical term  $t_i$  of the last sum. It equals to  $(x - y)^i / i! \cdot \sum_{k=i}^m (R_{a_{j+1,s}}^p f)^{(k)}(y)(y - a_{2j+2,s+1})^{k-i} / (k - i)!$ . Arguing as above, we obtain  $|t_i| \leq |x - y|^i / i! \cdot \varepsilon \sum_{k=i}^m l_s^{p-k} l_{s+1}^{k-i} / (k - i)! < e\varepsilon|x - y|^i l_s^{p-i} / i!$ . By that,  $|\sum_{i=0}^m t_i| \leq C_0^p e^2 \varepsilon|x - y|^p$ , which completes the proof.  $\square$

**Remarks.** 1. One can enumerate all functions from  $(e_{k,1,0})_{k=0}^\infty \cup (e_{k,2,j})_{k=0, j=1, s=1}^\infty$  and the corresponding functionals  $\eta$  into a biorthogonal sequence  $(f_n, \eta_n)_{n=1}^\infty$  in such way that for some increasing sequences  $(N_p)_{p=0}^\infty, (q_p)_{p=0}^\infty$  the sum  $S_{N_p}(f) = \sum_{n=1}^{N_p} \eta_n(f) \cdot f_n$  coincides with  $T_{a_{j,p}}^{q_p} f$  on  $I_{j,p}$  for  $1 \leq j \leq 2^p$ . Yet, the sequence  $(f_n, \eta_n)_{n=1}^\infty$  will not have the basis property in the space  $\mathcal{E}(K(\Lambda))$ . Indeed, let  $F \in C^\infty[0, 1]$  solve the Borel problem for the sequence  $(q_n ! l_n^{-q_n})_{n=0}^\infty$ , that is  $F^{(q_n)}(0) = q_n ! l_n^{-q_n}$  for  $n \in \mathbb{N}_0$  and  $F^{(k)}(0) = 0$  for  $k \neq q_n$ . Let



$f = F|_{K(\Lambda)}$ . Then  $|f - S_{N_p}(f)|_0 \geq |R_0^{q_p} f(l_p)| \geq \sum_{k=1}^{q_p} f^{(k)}(0) l_p^k / k! - |f(l_p) - f(0)| > 1 - |f(l_p) - f(0)|$ . The last expression has a limit 1 as  $p \rightarrow \infty$ , so  $S_N(f)$  does not converge to  $f$  in  $|\cdot|_0$ .

For a basis in the space  $\mathcal{E}(K(\Lambda))$ , see [6].

2. As concerns the paper by Jonsson [9], we note that natural triangulations of the set  $K(\Lambda)$  are given by the sequence  $\mathcal{F}_s = \{I_{i,s}, 1 \leq i \leq 2^s\}$ ,  $s \geq 0$ . The regularity conditions discussed in [9] are reduced in this case to (6) and

$$\liminf_{s \rightarrow \infty} \frac{l_{s+1}}{l_s} > 0. \quad (8)$$

Thus, provided these conditions, the expansion of  $f \in \mathcal{E}^p(K(\Lambda))$  with respect to Jonsson's interpolating system converges, at least in  $|\cdot|_p$ , to  $f$ , by Proposition 2 in [9]. It is interesting to check the corresponding convergence in topology given by the norm  $\|\cdot\|_p$ . At the same time it is essential for the proof of by Proposition 2 [9] that the diameters of neighboring triangulations are comparable, which is (8) for Cantor-type sets. Our construction can be applied to any “small” Cantor set with arbitrary fast decrease of the sequence  $(l_s)_{s=0}^\infty$ . The basis problem for the space  $\mathcal{E}^p(K(\Lambda))$  in the case of “large” Cantor set with  $l_s/h_s \rightarrow \infty$  is open.

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